The model also enables a treatment of extinction in terms of the kinetics of the propellant flame reactions, yielding a new form of equation for relating motor $L^*$ and extinction pressure which is in satisfactory agreement with the available data. However, it is clear that the existing data encompasses only a narrow part of the spectrum of experimental results required for an adequate test of this or any other theoretical treatment.

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# Circular Sandwich Plate under Radial Compression and Thermal Gradient

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SINCE the early work of Reissner[1] on sandwich construction, for which the effect of transverse shear deformation is known to be appreciable, the bending of circular sandwich plates has become the subject of numerous investigations. Ericksen[2] developed formulas for the deflection and stresses in a circular sandwich plate under normal load. A general treatment of the axisymmetric problem of bimetallic circular plates including thermal effects has been given by Grigoliuk.[3] Zaid[4] extended Reissner's work to include the bending rigidities of the facings. Thermal stresses in laminated circular plates have been investigated by Vinson.[5] More recently Bruun[6] has analyzed the thermal deflection problem of a circular sandwich plate. None of these works, however, has considered the combined effect of radial compression and a thermal gradient. Accordingly, the basic displacement equations in polar coordinates with axial symmetry derived elsewhere by Huang[7] for a circular sandwich plate (having a soft isotropic core and isotropic facings of different thicknesses and materials) under radial compression and thermal gradient and including the bending rigidity of the facings are here solved for certain special cases. These are

$$\left.\begin{aligned}
&\frac{E't'}{1-\mu'^2}\left(r\frac{d^2u'}{dr^2}+\frac{du'}{dr}-\frac{u}{r}\right)- \\
&\qquad r\bar{G}\left(\frac{u'-u''}{\bar{t}}+\hat{t}\frac{dW}{dr}\right)=r\frac{dN'}{dr} \\
&\frac{E''t''}{1-\mu''^2}\left(r\frac{d^2u''}{dr^2}+\frac{du''}{dr}-\frac{u''}{r}\right)+ \\
&\qquad r\bar{G}\left(\frac{u'-u''}{\bar{t}}+\hat{t}\frac{dw}{dr}\right)=r\frac{dN''}{dr}
\end{aligned}\right\} \tag{1}$$

$$\begin{aligned}
&D\nabla^2\nabla^2 w-\bar{t}\hat{t}\bar{G}\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{u'-u''}{\bar{t}}+\hat{t}\frac{dw}{dr}\right)\right]-q- \\
&\qquad N_r\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)=-\nabla^2 M
\end{aligned} \tag{2}$$

where, in addition to the quantities shown in Fig. 1,

$$\nabla^2\equiv\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\qquad N_r=N_r'+N_r''$$

$$M=M'+M''\qquad q=q'+q''$$

$$\begin{Bmatrix}N'\\M'\end{Bmatrix}=\frac{E'\alpha'}{1-\mu'}\int_{-t'/2}^{t'/2}T'\begin{Bmatrix}1\\z'\end{Bmatrix}dz'$$

$$\begin{Bmatrix}N''\\M''\end{Bmatrix}=\frac{E''\alpha''}{1-\mu''}\int_{-t''/2}^{t''/2}T''\begin{Bmatrix}1\\z''\end{Bmatrix}dz''$$

$$D=\frac{E't'^3}{12(1-\mu'^2)}+\frac{E''t''^3}{12(1-\mu''^2)}\qquad \hat{t}=\frac{\bar{t}+(t'+t'')/2}{\bar{t}}$$

and $u'$, $u''$, $N_r'$, $N_r''$, $q'$, $q''$, $E'$, $E''$, $\mu'$, $\mu''$, $\alpha'$, $\alpha''$, $T'$, and $T''$ denote, respectively, the radial displacements, radial membrane forces, external transverse normal loads, Young's moduli, Poisson's ratios, coefficients of thermal expansion, and temperature in the facings; $\bar{G}$ is the shear modulus of the core. The solution of Eq. (1) is

$$\left.\begin{aligned}
u'&=\frac{1-\mu'^2}{E't'\hat{t}\bar{t}}\left[D\frac{dw}{dr}+\tilde{N}_r\frac{1}{r}\int rwdr+\right. \\
&\quad\frac{1}{r}\int(M+\hat{t}\bar{t}N')rdr- \\
&\qquad\left.\frac{1}{r}\int rdr\int\frac{dr}{r}\int qrdr+\frac{c_2'r}{2}\right] \\
-u''&=\frac{1-\mu''^2}{E''t''\hat{t}\bar{t}}\left[D\frac{dw}{dr}+\tilde{N}_r\frac{1}{r}\int rwdr+\right. \\
&\quad\frac{1}{r}\int(M-\hat{t}\bar{t}N'')rdr- \\
&\qquad\left.\frac{1}{r}\int rdr\int\frac{dr}{r}\int qrdr+\frac{c_2''r}{2}\right]
\end{aligned}\right\} \tag{3}$$

where $\tilde{N}_r=-N_r$ and $C_2'$, $C_2''$ are integration constants the rest of which has been shown to vanish by Zaid.[4] Because of Eq. (3), but with $q=0$ and constant $N'$, $N''$, and $M$, Eq. (2) now becomes

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)w=N_1C_2/D \tag{4}$$

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where

$$\begin{Bmatrix} m_1^2 \\ m_2^2 \end{Bmatrix} = \frac{1}{2D} \{\pm N_1 D_0(\hat{D} - r_e k_e) \mp [N_1^2 D_0^2 (\hat{D} - r_e k_e)^2 + 4N_1 k_e P_e D]^{1/2}\}$$

$$N_1 = \frac{\bar{G}(1+m)(1-\mu'^2)}{E't'\bar{t}} \qquad D_0 = \frac{E't'\bar{t}^2\hat{t}^2}{(1+m)(1-\mu'^2)}$$

$$\hat{D} = \frac{D + D_0}{D_0} \qquad k_e = \frac{\bar{N}_r}{P_e}$$

$$r_e = \frac{P_e}{\bar{t}\hat{t}^2\bar{G}} \qquad P_e = \frac{4.2D_0}{a^2}$$

$$C_2 = M + \bar{C}_2 + \frac{M_0}{m} \qquad \bar{C}_2 = \frac{C_2' + mC_2''}{1+m}$$

$$m = \frac{h''}{h'} = \frac{E't'(1-\mu''^2)}{E''t''(1-\mu'^2)} \qquad M_0 = \frac{m\hat{t}\bar{t}}{1+m}(N' - mN'')$$

and $a$ is the radius of the sandwich plate. The general solution of Eq. (4) is

$$w = A_1 I_0(m_1 r) + B_1 K_0(m_1 r) + A_2 J_0(m_2 r) + B_2 Y_0(m_2 r) - \frac{N_1}{m_1 + m_2}\left[\int_{m_1\epsilon\to 0}^{m_1 r} C_2 \frac{K_0(s) I_0(m_1 r) - I_0(s) K_0(m_1 r)}{m_1^2 \{I_0(s)\dot{K}_0(s) - \dot{I}_0(s) K_0(s)\}} ds + \int_{m_2\epsilon\to 0}^{m_2 r} C_2 \frac{Y_0(s) J_0(m_2 r) - J_0(s) Y_0(m_2 r)}{m_2^2 \{J_0(s)\dot{Y}_0(s) - \dot{J}_0(s) Y_0(s)\}} ds\right] \quad (5)$$

where $I_0$, $K_0$, $J_0$, $Y_0$ are Bessel functions, and a dot denotes differentiation with respect to the integration variable $s$. The determination of the constants of integration has been illustrated by Zaid.[4]

For a simply supported sandwich plate and $\mu' = \mu'' = \mu$, Eqs. (3) and (5) become

$$\bar{u} = \frac{mu' + u''}{1+m} = \frac{m(1-\mu)}{(1+m)E't'}(-\bar{N}_r + N)r$$

$$u = \frac{m(\mu' - \mu'')}{1+m} = \frac{1-\mu^2}{E't'\bar{t}}\left\{D\frac{dw}{dr} + \frac{[M\bar{N}_r + Dm_1^2(M_0 + M)]I_1(m_1 r)}{Dm_1(m_1^2 + m_2^2)[\ddot{I}_0(m_1 a) + (\mu/m_1 a)\dot{I}_0(m_1 a)]} - \frac{[M\bar{N}_r + Dm_2^2(M_0 + M)]J_1(m_2 r)}{Dm_2(m_1^2 + m_2^2)[\ddot{J}_0(m_2 a) + (\mu/m_2 a)\dot{J}_0(m_2 a)]}\right\}$$

$$w = -\frac{[M\bar{N}_r + Dm_1^2(M_0 + M)][I_0(m_1 r) - I_0(m_1 a)]}{\bar{N}_r D(m_1^2 - m_2^2)[\ddot{I}_0(m_1 a) + (\mu/m_1 a)\dot{I}_0(m_1 a)]} + \frac{[M\bar{N}_r + Dm_2^2(M_0 + M)][J_0(m_2 r) - J_0(m_2 a)]}{\bar{N}_r D(m_1^2 + m_2^2)[\ddot{J}_0(m_2 a) + (\mu/m_2 a)\dot{J}_0(m_2 a)]}$$

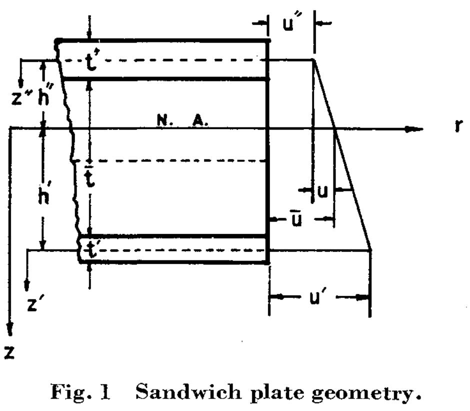


**Fig. 1 Sandwich plate geometry.**

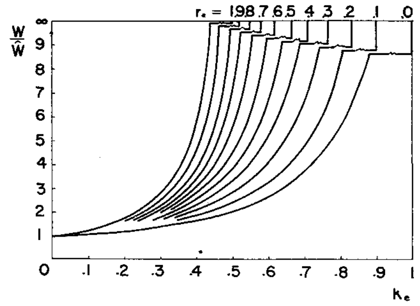


**Fig. 2 Ratio of maximum deflection to maximum thermal deflection ($W/\hat{W}$) vs edge loading parameter $k_e$ for different values of $r_e$.**

where $\bar{u}$ and $u$, following the notation of Ebcioglu,[8] are shown in Fig. 1.

When the bending rigidity of the facings is neglected and the temperature distribution is such that $M = 0$, the transverse normal deflection is given by

$$w = \frac{M_0}{k_e P_e}\left[\frac{J_0(m_2 r) - J_0(m_2 a)}{\ddot{J}_0(m_2 a) + (\mu/m_2 a)\dot{J}_0(m_2 a)}\right]$$

We record the critical value of the radial compressive load for $\mu = 0.3$ and tabulated values of $J_0$ and $J_1$ as

$$(\bar{N}_r)_{cr} = P_e/(1 + P_e/\bar{t}\hat{t}^2\bar{G})$$

The ratio of maximum deflection $W$ to maximum thermal deflection $\hat{W}$ at $r = 0$, is found to be

$$\frac{W}{\hat{W}} = \frac{1+\mu}{2.1\,k_e}\left[\frac{1 - J_0(2.05\lambda)}{J_0(2.05\lambda) - \dfrac{1-\mu}{2.05\lambda}J_1(2.05\lambda)}\right]$$

where

$$\lambda = \left(\frac{k_e}{1 - r_e k_e}\right)^{1/2} \qquad r_e = \frac{1}{k_e} - 1 \qquad \left(\frac{P_e}{D_0}\right)^{1/2} = \frac{2.05}{a}$$

This ratio is plotted in Fig. 2 against the edge loading parameter $k_e$ for various values of $r_e$. We note that $k_e \to 0$ as $W/\hat{W} \to 1$ for a rigid, i.e., $r_e = 0$, core. As $k_e \to 1$, $W/\hat{W} \to \infty$, i.e., buckling occurs if the plate is still elastic. The critical load decreases as $r_e$ increases.

We conclude by further noting that multiplication of Eq. (4) by $\nabla^2$ leads to

$$D\nabla^2\nabla^2\nabla^2 w - N_1[D_0(\hat{D} - k_e r_e)\nabla^2\nabla^2 w + k_e P_e \nabla^2 w] = 0 \quad (6)$$

Letting $\bar{G} \to 0$, for a finite $\bar{t}$, we obtain from Eq. (6)

$$(D + D_0)\nabla^2\nabla^2 w = \bar{N}_r \nabla^2 w$$

a result that coincides with that of classical plate theory. A similar conclusion is also reached by letting $\bar{t} \to 0$ in Eq. (6) as, once again, the effect of the core is made to vanish.

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# Simplified Solutions for Ablation in a Finite Slab

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### Nomenclature

$B_1$ to $B_4$ = constants of integration
$c$ = specific heat, Btu/lb-°F
$h$ = combined heat-transfer coefficient, Btu/sec-°F-in.$^2$ $= h_c + h_r$
$h_c$ = convective heat-transfer coefficient, Btu/sec-°F-in.$^2$
$h_r$ = radiative heat-transfer coefficient, Btu/sec-°F-in.$^2$
$\Delta H_{eff}$ = effective heat of ablation $= L + c(T_0 - T_m)\rho_c/\rho_s$, Btu/lb
$k$ = thermal conductivity, Btu/sec-in.-°F
$L$ = heat of charring, Btu/lb
$s$ = distance of the ablative material being removed away which is function of time, in.
$T$ = temperature, °F
$v$ = ablation velocity, in./sec
$x$ = distance coordinate in direction of ablation, in.
$\alpha$ = thermal diffusivity $= k/c\rho$, in.$^2$/sec
$\epsilon$ = porosity of the ablative material
$\rho$ = density, lb/in.$^3$
$\delta$ = depth, in.
$\xi$ = distance coordinate $= x - s(\theta)$ (see Fig. 1)
$\theta$ = time, sec
$\lambda$ = proportional constant defined by Eq. (19)
erf = error function

***Subscripts***

$c$ = char-gas layer
$s$ = virgin material
$i$ = initial
0 = surface
$g$ = recovery gas
$m$ = ablation

IN a rocket motor, when heat flows from the exhaust gases to the ablating material, there are two heating periods. The first period raises the surface temperature of the material to the ablation temperature The time required to reach this temperature can be estimated by the method described in Ref. 1. After the surface reaches this temperature, the second period is initiated during which the ablation penetrates into the material at this temperature. Some of the material is continuously removed by being blown away, some becomes charred, and some is unchanged. The model is shown in Fig. 1. In order to simplify the problem, the following assumptions are made: 1) flow is one-dimensional and linear, 2) thermal equilibrium is established in the char-gas layer, 3) heat of reaction of the ablation material is negligible, 4) porosity of the ablative material is negligible, 5) the material is opaque, 6) temperature gradient in the virgin material is negligible [to be used in Eq. (10)], 7) transpiration and diffusion effects of the gases are negligible, and 8) thermal conductivity and thermal diffusivity of the material and the combined heat-transfer coefficient are constant.

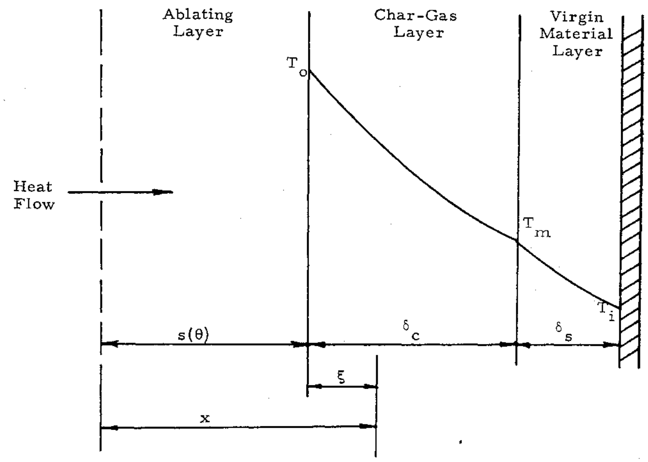


**Fig. 1 Ablation model.**

The one-dimensional heat conduction equation is

$$\partial T/\partial\theta = \alpha\partial^2T/\partial x^2 \tag{1}$$

Figure 1 shows that

$$\xi = x - s(\theta) \tag{2}$$

where $s$ is a function of time $\theta$. Equation (1) can be transformed to this variable as

$$\partial T/\partial\theta = \alpha(\partial^2T/\partial\xi^2) + ds/d\theta\,\partial T/\partial\xi \tag{3}$$

which is the equation for the gas-char layer. The virgin material layer is governed by Eq. (1). The initial and boundary conditions are

$$T_c(x, 0) = T_s(x, 0) = T_i \tag{4}$$

$$T_c(0, \theta) = T_0 \tag{5}$$

$$T_c(\delta_c, \theta) = T_m \tag{6}$$

$$T_s(\delta_c, \theta) = T_m \tag{7}$$

$$T_s(\delta_s, \theta) = T_i \tag{8}$$

In addition, the heat balances give two more boundary conditions:

$$\text{at } \xi = 0 \qquad -k_c(\partial T/\partial\xi) = h(T_g - T_0) \tag{9}$$

$$\text{at } \xi = \delta_c \qquad k_s(\partial T_s/\partial\xi) - k_c(\partial T_c/\partial\xi) = L\rho_s(ds/d\theta) \tag{10}$$

Because of assumption (6), Eq. (10) becomes

$$\text{at } \xi = \delta_c \qquad -k_c(\partial T_c/\partial\xi) = L\rho_s(ds/d\theta) \tag{11}$$

where $L$ may be defined as the heat of charring, which is analogous to the heat of melting.

### Steady-State Solution

***Char-gas layer***

Equation (3) becomes

$$\alpha\partial^2T/\partial\xi^2 + v\partial T/\partial\xi = 0 \tag{12}$$

where $v$ is the ablation velocity that does not vary with time. The initial and boundary conditions are Eqs. (4–6 and 9) and

$$-k_c(\partial T_c/\partial\xi)_{\delta c} = L\rho_s v \tag{13}$$

which is a form of Eq. (11) under steady-state conditions.

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